Semester V (U.G.) 03.11.2025

MJC 08

Ring Theory (continued)

We are now familiar with the definition of a ring $(R, +, \cdot)$. In general, we simply say that R is a ring $\Rightarrow (R, +, \cdot)$ is a ring.

Now, we define some types of rings.

(I) <u>Commutative ring</u> : If multiplication on R is commutative, i.e. $ab = ba \ \forall \ a, b \in R$

(II) <u>Ring with unity</u> If the element $1 \in R$ such that 1 is the multiplicative identity called as <u>unity of R</u> denoted by 1.

(III) <u>Ring with zero divisors</u>

If there exists at least two non-zero elements a and b of R such that $ab = 0$ but $a \neq 0, b \neq 0$.

IV <u>Ring without zero divisors</u>, i.e.

$$ab = 0 \Leftrightarrow a = 0 \text{ or } b = 0.$$

Theorem

In a ring $(R, +, \cdot)$ prove that if $a, b, c \in R$ then

(i) $a.0 = 0.a = 0$ where 0 is the additive identity of R

(ii) $a.(-b) = (-a).b = -(ab)$ $\forall a, b$

(iii) $(-a).(-b) = ab$

Proof

(i) 0 is additive identity of R.

$\Rightarrow$ $0 + 0 = 0$ ——— (1)

$\therefore$ $a.0 = a(0+0)$ using (1)

$\Rightarrow$ $a.0 = a.0 + a.0$. [By distributive law]

$\Rightarrow$ $a.0 + 0 = a.0 + a.0$. [By property of 0]

using left cancellation law, we get

$0 = a.0$ i.e. $a.0 = 0$ Proved

Similarly we can show that $0.a = 0$

$\Rightarrow$ $a.0 = 0.a = 0$.

(ii) we know, by property of R, that

$b + (-b) = (-b) + b = 0$ ——— (2)

Now, $a.0 = 0$

$\Rightarrow a.[b + (-b)] = 0$ [using (2)]

$\Rightarrow \quad a \cdot b + a \cdot (-b) = 0$

$\Rightarrow \quad a \cdot (-b) = -(a \cdot b) \quad \text{———} \ (3)$

Again $\quad a, b \in R \Rightarrow a \cdot b \in R \Rightarrow -(a \cdot b) \in R$

Also, $b \in R \Rightarrow -b \in R$

$\Rightarrow [(-a) + a] = 0$

$\therefore \cancel{(a \cdot b) + (a \cdot b) = 0} \longrightarrow$ [By inverse law on (+)]

$\cancel{\Rightarrow (a \cdot b) - (a \cdot b) =}$

$\Rightarrow [(-a) + a]\, b = 0 \cdot b = 0$

$\Rightarrow (-a) \cdot b + ab = 0$

$\Rightarrow (-a) b = -(a \cdot b) \quad \text{———} \ (4)$

From (3) and (4), we have

$a \cdot (-b) = (-a) \cdot b = -(a \cdot b) \quad \text{———} \ (5)$

(iii) (a) $\because a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

$\Rightarrow (-a) \cdot (-b) = -[a \cdot (-b)]$

$= -[-(a \cdot b)]$ [using (5)]

$= (a \cdot b)$

Hence $(-a) \cdot (-b) = a \cdot b$